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Benefit and Distance Functions

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We explore the relationship between R. W. Shephard's input distance function ("Cost and Production Functions," Princeton Univ. Press, Princeton, 1953) and D. G. Luenberger's benefit function (*J. Math. Econ.* **21** (1992a), 461–481). We point out that the latter can be recognized in a production context as a directional input distance function which can exhaustively characterize technologies in both price and input space. D. McFadden's (Cost, revenue, and profit functions, *in* "Production Economics: A Dual Approach to Theory and Applications, "North-Holland/ Elsevier, New York, 1978) composition rules for input sets and input distance functions are then extended to the directional input distance function. *Journal of Economic Literature Classification Numbers*: D21, D24, D29. © 1996 Academic Press, Inc.

In a sequence of publications Luenberger [10–14] has introduced and applied a function he terms the benefit function[1], which is a directional representation of preferences. If $u(x)$ is a utility function, $x \in X \subset \mathfrak{R}^N_+$, and g a vector in $\mathfrak{R}^N_+$, then the benefit function is defined by

$$b(g; u, x) = \sup_{\beta} \{\beta \in \mathfrak{R}\colon x - \beta g \in X,\ u(x - \beta g) \geqslant u\}.$$

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[1] Luenberger [11, p. 148] traces it back to Dupuit [5]. Blackorby and Donaldson [1] employ a version of the benefit function, which they call the translation function, in their study of absolute inequality measurement.

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The benefit function can be recognized as a generalization of Shephard's [17] input distance function, which when defined in terms of the utility function (Deaton [4]) can be written

$$D_i(u, x) = \inf\{\lambda\colon (x/\lambda) \in X,\ u(x/\lambda) \geqslant u\}.$$

Both functions are useful alternative representations of preferences that may be exploited to advance different theoretical and empirical objectives. To wit, Luenberger [10, p. 480] states "The distance function can be useful in developing relations in individual consumer theory. The benefit function has use in developing group welfare relations."

This paper explores the relationship between the two functions and shows how each can be derived from the other. We then discuss duality theorems, shadow prices, and composition rules for the two functions.

1. DISTANCE FUNCTIONS

Although Luenberger has developed the benefit function as a tool in consumer theory, here we will study it as a tool in production theory.[2] Let $y \in \Re^M_+$ be a vector of outputs and $x \in \Re^N_+$ a vector of inputs. The technology is represented by input correspondences $L\colon \Re^M_+ \to \Re^N_+$ which define input sets $L(y) \subset \Re^N_+$:

$$L(y) = \{x\colon x \text{ can produce } y\}, \qquad y \in \Re^M_+. \tag{1.1}$$

Following Shephard [18] or Färe [6], define the input distance function by

$$D_i(y, x) = \sup\{\lambda \in \Re_+\colon (x/\lambda) \in L(y)\}. \tag{1.2}$$

This function inherits the properties imposed on the technology, see Shephard [18] or Färe [6], and in particular, assuming weak disposability of inputs, i.e., $x \in L(y) \Rightarrow \lambda x \in L(y)$, for $\lambda \geqslant 1$,

$$D_i(y, x) \geqslant 1 \text{ if and only if } x \in L(y), \tag{1.3}$$

which shows that the input distance function is a complete function representation of the technology. Hence, conditions on the technology can be equivalently expressed in terms of the input distance function or the input set. Moreover, from its definition it follows that the distance function is homogeneous of degree $+1$ in inputs. This property has proved especially useful in the construction of index numbers using distance functions (Malmquist [16], and Caves *et al.* [2]).

[2] Luenberger introduced the notion of a shortage function in his studies of production.

To motivate our terminology, let us first recall the notion of a direction. Let x and g be fixed vectors in $\Re^n$, then

$$z = x + \beta g, \qquad \beta \in \Re \tag{1.4}$$

defines a line in *the direction* of g. Clearly the benefit function may also be thought of as a directional concept. Thus we define $\vec{D}_i: \Re^M_+ \times \Re^N_+ \times \Re^N \to \Re$ by

$$\begin{aligned} \vec{D}_i(y, x; g) &= \sup_\beta \{\beta \in \Re: x - \beta g \in L(y)\}, \\ &= \sup\{\beta \in \Re: x \in \beta g + L(y)\} \end{aligned} \tag{1.5}$$

as the *directional input distance function.*[3] It is, of course, the benefit function for $L(y)$ as defined by Luenberger. Moreover, the second equality shows that the benefit function or directional input distance function is the maximal translation of $L(y)$ along g that permits keeping x feasible.

Three special cases are illustrated in Fig. 1a, 1b, and 1c. In Fig. 1a, $x \in L(y)$ and $\vec{D}_i(y, x; g)$ is given by the ratio $\|g^*\|/\|g\| > 0$. In Fig. 1b, $x \notin L(y)$ but moving x in the direction of g eventually encounters $L(y)$. Here $\vec{D}_i(y, x; g) = -\|g^*\|/\|g\| < 0$. Figure 1c illustrates the case where moving x in the direction of g never encounters $L(y)$, and thus $\vec{D}_i(y, x; g) = -\infty$.

The basic properties of the directional input distance function are summarized by the following lemma which slightly expands results due to Luenberger [10] (all proofs are in the Appendix).

(1.6) LEMMA. *$\vec{D}_i: \Re^M_+ \times \Re^N_+ \times \Re^N_+ \to \Re$ satisfies:*

(1) *if $L(y)$ is convex for all $y \in \Re^M_+$, $\vec{D}_i(y, x; g)$ is concave with respect to x;*

(2) *$\vec{D}_i(y, x + \alpha g; g) = \vec{D}_i(y, x; g) + \alpha$ for $\alpha \in \Re_+$;*

(3) *$x \in L(y)$ implies that $\vec{D}_i(y, x; g) \geqslant 0$;*

(4) *$\vec{D}_i(y, x; \mu g) = (1/\mu)\, \vec{D}_i(y, x; g)$, for $\mu > 0$.*

(5) (a) *if $y' \geqslant y \Rightarrow L(y') \subset L(y)$, then $y' \geqslant y \Rightarrow \vec{D}_i(y', x; g) \leqslant \vec{D}_i(y, x; g)$;*

(b) *if $L(y) \subset L(\lambda y)$, $0 < \lambda < 1$, then $\vec{D}_i(y, x; g) \leqslant \vec{D}_i(\lambda y, x; g)$;*

(6) *if $x \in L(y) \Rightarrow \lambda x \in L(y)$ for $\lambda > 1$, then $\vec{D}_i(y, \lambda x; g) \geqslant \lambda \vec{D}_i(y, x; g) = \vec{D}_i(y, x; g/\lambda)$.*

[3] Directional output distance functions and undesirable outputs are studied by Chung [3]

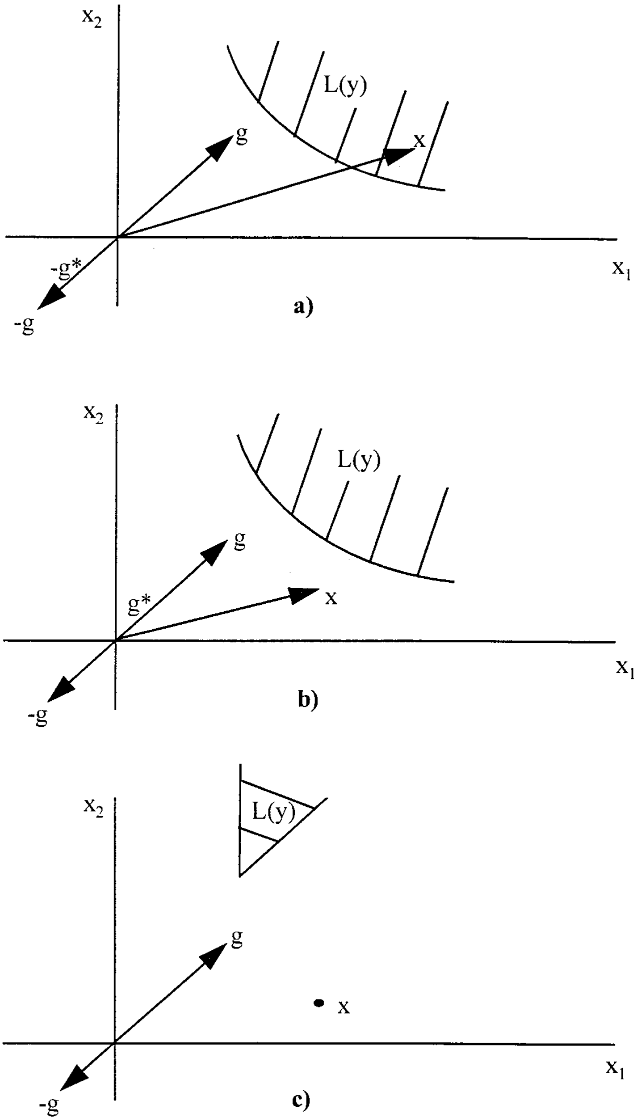

FIG. 1. Directional input distance functions.

Because $D_i(y, x) \geqslant 1$ if and only if $x \in L(y)$ under weak input disposability, the directional distance function can also be defined as

$$\vec{D}_i(y, x; g) = \sup\{\beta \in \Re: D_i(y, x - \beta g) \geqslant 1\}. \tag{1.7}$$

Expression (1.7) shows that the directional input distance function can be obtained from the input distance function. We now show that by an appropriate choice of g, we can always recover $D_i(y, x)$ and hence $L(y)$ from $\vec{D}_i(y, x; g)$. In particular, for $g = x$,

$$\begin{aligned} \vec{D}_i(y, x; x) &= \sup\{\beta \in \Re: x(1-\beta) \in L(y)\} \\ &= 1 - \inf\{(1-\beta): x(1-\beta) \in L(y), \beta \in \Re\} \\ &= 1 - \inf\{(1-\beta) \in R_+ : x(1-\beta) \in L(y), \beta \in \Re\} \\ &= 1 - 1/D_i(y, x), \end{aligned} \tag{1.8}$$

where the third equality follows from the fact that $L(y) \subset \Re^N_+$. By a similar argument, it follows that $D_i(y, x) = 1/\vec{D}_i(y, 0; -x)$. Using (1.3), it is immediate from (1.8) that under weak disposability of inputs,

$$\vec{D}_i(y, x; x) \geqslant 0 \Leftrightarrow x \in L(y). \tag{1.9}$$

Under free input disposability, i.e., $x' \geqslant x \in L(y) \Rightarrow x' \in L(y)$, expression (1.9) for $g \in \Re^N_+$ can be strengthened to

$$\vec{D}_i(y, x; g) \geqslant 0 \Leftrightarrow x \in L(y). \tag{1.9$'$}$$

That $\vec{D}_i(y, x; g) \geqslant 0$ for $x \in L(y)$ follows immediately from (1.6.3). To prove the converse, suppose first that $\vec{D}_i(y, x; g)$ equals zero; it is then immediate that $x \in L(y)$. Now suppose that $\vec{D}_i(y, x; g) > 0$, and note that $x \geqslant x - \vec{D}_i(y, x; g) g \in L(y)$ which yields $x \in L(y)$ under free disposability. Under appropriate disposability assumptions, $\vec{D}_i(y, x; g)$ is a complete function representation of $L(y)$.

A third input distance function, the affine distance function, has been introduced by Färe and Lovell [8]. This distance function is defined as

$$\begin{aligned} D_i^0(y, x; x^0) &= [\inf\{\lambda: x^0 + \lambda x \in L(y)\}]^{-1} = [\inf\{\lambda: \lambda x \in L(y) - x^0\}]^{-1} \\ &= [\inf\{\lambda: D_i(y, x^0 + \lambda x) \geqslant 1\}]^{-1}. \end{aligned} \tag{1.10}$$

If $x = x^0$, then

$$\begin{aligned} D_i^0(y, x; x) &= [\inf\{\lambda: D_i(y, x + \lambda x) \geqslant 1\}]^{-1} \\ &= D_i(y, x)/(1 - D_i(y, x)), \end{aligned} \tag{1.11}$$

or equivalently

$$D_i(y, x) = D_i^0(y, x; x)/(1 + D_i^0(y, x; x)). \tag{1.12}$$

Expressions (1.9) and (1.10) show one relationship between Shephard's input distance function and the affine distance function. Of course if $x^0 = 0$ then the two distance functions are equal. Note also that the affine and the directional distance functions are related by

$$\vec{D}_i(y, x^0; -x) = 1/D_i^0(y, x; x^0), \tag{1.13}$$

or since $\vec{D}_i(y, x; g)$ is homogeneous of degree -1 in g,

$$\vec{D}_i(y, x^0; x)\, D_i^0(y, x; x^0) = -1. \tag{1.14}$$

Summarizing the preceding, we have:

(1.15) PROPOSITION. *Let* $D_i(y, x)$, $\vec{D}_i(y, x; g)$, and $D_i^0(y, x; x^0)$ denote the input distance function, the directional input distance function, and the affine input distance function, respectively. If $x \in L(y) \Rightarrow \lambda x \in L(y)$ for $\lambda > 1$, then:

(1) $\vec{D}_{\mathrm{i}}(y, x; x) \geqslant 0 \Leftrightarrow x \in L(y)$;
(2) $\vec{D}_i(y, x; x) = 1 - 1/D_i(y, x)$; and
(3) $\vec{D}_{\mathrm{i}}(y, x^0; -x) = 1/D_1^0(y, x; x^0)$.

If $x' \geqslant x \in L(y) \Rightarrow x' \in L(y)$, then

(4) $\vec{D}_i(y, x; g) \geqslant 0 \Leftrightarrow x \in L(y)$ for $g \in \Re_+^N$.

2. DUALITIES AND SHADOW PRICES

The directional and the usual distance functions are both primal representations of the technology, expressed by input requirements sets $L(y)$, $y \in \Re_+^M$. The dual representation of the technology is, of course, given by the cost function. Let $w \in \Re_+^N$ denote a vector of input prices; the cost function (or the expenditure function in consumer theory) is then given by

$$C(y, w) = \inf_x \{wx\colon x \in L(y)\}, \qquad y \in \Re_+^M \tag{2.1}$$

or equivalently

$$C(y, w) = \inf_x \{wx\colon D_i(y, x) \geqslant 1\}, \qquad y \in \Re_+^M. \tag{2.2}$$

Shephard [17, 18] proved that if the input correspondence, L, satisfies (we always presume these hold in what follows) $L(0)=\Re^N_+$, $0 \notin L(y)$ for $y \geqslant 0$, $y \neq 0$; L is a closed correspondence; free input disposability; and $L(y)$ is convex for all $y \in \Re^M_+$, then the cost and the input distance functions are dual to each other:

$$C(y, w)=\inf_x \{wx: D_i(y, x) \geqslant 1\}, \qquad y \in \Re^M_+ \tag{2.3}$$

$$D_i(y, x)=\inf_w \{wx: C(y, w) \geqslant 1\}, \qquad y \in \Re^M_+. \tag{2.4}$$

From (1.8) it now follows immediately that under weak input disposability

$$C(y, w)=\inf_x \{wx: \vec{D}_i(y, x; x) \geqslant 0\}$$

and

$$\vec{D}_i(y, x; x)=1-1/\inf_w \{wx: C(y, w) \geqslant 1\}.$$

Shephard's duality theorem can also be expressed as a pair of unconstrained optimization problems, namely,[4]

$$C(y, w)=\inf_x \{wx/D_i(y, x)\}, y \in \Re^M_+, \tag{2.5}$$

$$D_i(y, x)=\inf_w \{wx/C(y, w)\}, y \in \Re^M_+. \tag{2.6}$$

Luenberger [10, 14] proves, under the same conditions as above, a duality theorem between goods and goods prices. His theorem can be stated in our terminology as

$$C(y, w)=\inf_x \{wx-\vec{D}_i(y, x; g)\, wg\} \tag{2.7}$$

$$\vec{D}_i(y, x; g)=\inf_w \{wx-C(y, w): wg=1\}. \tag{2.8}$$

This duality theorem like Shephard's shows that inputs x and deflated input prices w are dual. The Luenberger duality theorem can also be expressed as a pair of unconstrained optimization problems, namely,

$$C(y, w)=\inf_x \{wx-\vec{D}_i(y, x; g) \cdot wg\} \tag{2.9}$$

$$\vec{D}_i(y, x; g)=\inf_w \left\{\frac{wx-C(y, w)}{wg}\right\}. \tag{2.10}$$

[4] See Färe and Primont [9].

By choosing $g=x$, expression (2.9) reduces to (2.5), and (2.10) reduces to (2.6), showing that the Färe and Primont formulation of Shephard's (input) duality theorem is a consequence of the Luenberger duality theorem.

Färe and Grosskopf [7] developed a dual Shephard's lemma and showed that virtual prices can be derived from the distance function. Assuming that the cost and input distance functions are both differentiable, they showed that

$$w_n = C(y, w)\,\frac{\partial D_i(y, x)}{\partial x_n}, \qquad n=1, ..., N. \tag{2.11}$$

This result follows from applying the envelope theorem to expression (2.3). In the same way, by applying the envelope theorem to (2.7), the adjusted price function of Luenberger is derived, namely

$$w_n = wg\,\frac{\partial \vec{D}_i(y, x; g)}{\partial x_n}, \qquad n=1, ..., N. \tag{2.12}$$

The last two expressions are two variations on the dual Shephard's lemma. Their difference consists of the different scaling factors, $C(y, w)$ versus wg.

3. COMPOSITION RULES FOR DIRECTIONAL DISTANCE FUNCTION

McFadden [15] has presented a tabular representation of equivalent structural restrictions on the technology stated in dual and primal terms. Our next result extends McFadden's existing composition rules on input sets to the directional input distance function.

(3.1) Proposition. *Let* $L^j(y) \subset \Re^N_+$ $(j=1, ..., J)$ represent convex input sets satisfying free disposability of inputs, $\vec{D}^j_i(y, x; g)$ $(j=1, ..., J)$ the corresponding directional input distance functions, $\vec{D}^{zj}_i(y, x; g)$ the directional input distance function for $z_j L^j(y)$ where $z_j \in \Re_+$, and $L^*(y)$ a convex input set satisfying free disposability of inputs: Then the following pairs of representations of the technology are equivalent:

(a) $\quad L^0(y) = a(y)\, L^1(y) \quad$ where $\quad a\colon \Re^M_+ \to \Re_{++}$,

(a′) $\quad \vec{D}^0_i(y, x; g) = \vec{D}^1_i\left(y, \dfrac{x}{a(y)}, \dfrac{g}{a(y)}\right) = a(y)\, \vec{D}^1_i\left(y, \dfrac{x}{a(y)}, g\right),$

(b) $L^0(y)=\{A(y)x\colon x\in L^1(y)\}$ where $A(y)$ is a diagonal $N\times N$ matrix with strictly positive diagonal elements,

$$\text{(b}'\text{)} \qquad \vec{D}_i^0(y, x; g)=\vec{D}_i^1(y, A^{-1}(y)\,x, A^{-1}(y)g),$$

$$\text{(c)} \qquad L^0(y)=\sum_{j=1}^{J} L^j(y),$$

$$\text{(c}'\text{)} \qquad \vec{D}_i^0(y, x; g)=\sup\left\{\min_{j=1,\dots,J}\{\vec{D}_i^j(y, x^j, g^j)\}\colon x^j\geqslant 0,\ \sum_{j=1}^{J} x^j=x;\ \sum_{j=1}^{J} g^j=g\right\},$$

$$\text{(d)} \qquad L^0(y)=\bigcap_{j=1}^{J} L^j(y),$$

$$\text{(d}'\text{)} \qquad \vec{D}_i(y, x; g)=\min_{j=1,\dots,J}\{\vec{D}_i^j(y, x; g)\},$$

$$\text{(e)} \qquad L^0(y)=\text{convex hull of } \bigcup_{j=1}^{J} L^j(y),$$

$$\text{(e}'\text{)} \qquad \vec{D}_i^0(y, x; g)=\sup\left\{\min_{j=1,\dots,J}\{\vec{D}_i^{zj}(y, x^j, g^j)\}\colon \sum_{j=1}^{J} x^j=x,\ \sum_{j=1}^{J} g^j=g,\ z_j\in\Re_+ (j=1, \dots, J),\ \sum_{j=1}^{J} z_j=1\right\},$$

$$\text{(f)} \qquad L^0(y)=\bigcup_{\substack{z_j\in\Re_+\\ \sum_{j=1}^{J} z_j=1}}\ \bigcap_{j=1}^{J} z_j L^j(y)$$

$$\text{(f}'\text{)} \qquad \vec{D}_i^0(y, x; g)=\sup_z\left\{\min_{j=1,\dots,J}\{\vec{D}_i^{zj}(y, x, g)\}\colon \sum_{j=1}^{J} z_j=1,\ z_j\in\Re_+, j=1, \dots, J\right\},$$

(g) $$L^0(y) = \bigcup_{z \in L^*(y)} \sum_{j=1}^{J} z_j L^j(y),$$

(g′) $$\vec{D}_i^0(y, x; g)$$
$$= \sup \left\{ \min_{j=1,\ldots,J} \{\vec{D}_i^{zj}(y, x^j, g^j)\} : x^j \geqslant 0, \sum x^j = x, \sum_{j=1}^{J} g^j = g, z \in L^*(y) \right\},$$

(h) $$L^0(y) = \text{closure} \left\{ \bigcup_{z \in L^*(y)} \bigcap_{j=1}^{J} z_j L^j(y) \right\},$$

(h′) $$\vec{D}_i^0(y, x; g) = \sup_{z \in L^*(y)} \left\{ \min_{j=1,\ldots,J} \{\vec{D}_i^{zj}(y, x; g)\} \right\}.$$

APPENDIX

Proof of (1.6) *Lemma.* (1), (2), and (3) are proven in Luenberger [10]. If $y' \geqslant y \Rightarrow L(y') \subset L(y)$ then $x - \vec{D}_i(y', x; g)\, g \in L(y)$; hence, $\vec{D}_i(y, x; g) \geqslant \vec{D}_i(y', x; g)$ which shows (5a), (5b) follows similarly. To establish (4), note that

$$\vec{D}_i(y, x; \mu g) = \sup\{\beta \in \Re: x - \beta\mu g \in L(y)\}$$
$$= \frac{1}{\mu} \sup\{\beta\mu \in \Re: x - \beta\mu g \in L(y)\}$$
$$= \frac{1}{\mu} \vec{D}_i(y, x, g).$$

If $x \in L(y) \Rightarrow \lambda x \in L(y)$, for $\lambda \geqslant 1$, then $\lambda(x - \vec{D}_i(y, x; g)\, g) \in L(y)$ whence $\vec{D}_i(y, \lambda x; g) \geqslant \lambda \vec{D}_i(y, x; g) = \vec{D}_i(y, x; g/\lambda)$ by (1.6.4), which establishes (6). Q.E.D.

Proof of (3.1) *Proposition.* We only prove ( ) ⇒ (′) in each case. The converse is straightforward upon using (1.9′) and is left to the reader.

(a) implies that

$$\vec{D}_i^0(y, x, g) = \sup\{\beta \in \Re: x - \beta g \in L^0(y)\}$$
$$= \sup\{\beta \in \Re: x - \beta g \in a(y)\, L^1(y)\}$$

$$= \sup \left\{ \beta \in \mathfrak{R}: \frac{x}{a(y)} - \beta \frac{g}{a(y)} \in L^1(y) \right\}$$

$$= \vec{D}_i^1 \left(y, \frac{x}{a(y)}, \frac{g}{a(y)} \right)$$

$$= a(y)\, \vec{D}_i^1 \left(y, \frac{x}{a(y)}, g \right),$$

where the last equality follows by (1.6.4). This establishes that (a) $\Rightarrow$ (a′); (b) and (b′) are derived similarly. By (c)

$$L^0(y) = \sum_{j=1}^{J} L^j(y),$$

and thus

$$x - \beta g \in L^0(y)$$

implies that there exist nonnegative allocations x^j, g^j such that

$$x^j - \beta g^j \in L^j(y),$$

$j = 1, ..., J$, from which it immediately follows that

$$\beta \leqslant \vec{D}_i^j(y, x^j, g^j) \qquad j = 1, ..., J$$

so that for any arbitrary nonnegative allocation x^j, g^j such that $\sum_{j=1}^J x^j = x$, $\sum_{j=1}^J g^j = g$, the largest that such a β can be is

$$\min_{j=1, ..., J} \{ \vec{D}_i^j(y, x^j, g^j) \},$$

where the preceding notation denotes the minimum over the set of directional input distance functions. Hence,

$$\vec{D}_i^0(y, x; g) = \sup \left\{ \min_{j=1, ... J} \{ \vec{D}_i^j(y, x^j, g^j) \}: \sum_{j=1}^{J} g^j = g, \sum_{j=1}^{J} x^j \in \mathfrak{R}_+^N, j = 1, ..., J \right\}.$$

By (d)

$$x - \vec{D}_i^0(y, x; g)\, g \in L^j(y)$$

for $j=1, ..., J$. Hence,

$$\vec{D}_i^0(y, x; g) \leqslant \vec{D}_i^j(y, x; g)$$

whence

$$\vec{D}_i^0(y, x; g) = \min_{j=1, ..., J} \{\vec{D}_i^j(y, x; g)\}.$$

By (e) $L^0(y) =$ convex hull of $\bigcup_{j=1}^J L^j(y)$ which implies that

$$L^0(y) = \left\{x\colon x \in \sum_{j=1}^J z_j L^j(y), z_j \in \Re_+, \sum_{j=1}^J z_j = 1\right\}.$$

Hence, if $x - \beta g \in L^0(y)$, there must exist nonnegative allocations x^j, g^j with $\sum_{j=1}^J x^j = x$, $\sum_{j=1}^J g^j = g$, such that

$$x^j - \beta g^j \in z_j L^j(y)$$

for some set of z_j's. Recall that

$$\vec{D}_i^{zj}(y, x^j, g^j) = \sup\{\beta \in \Re\colon x^j - \beta g^j \in z_j L^j(y)\}.$$

Now follow the proof of (c′) to obtain (e′).

From (3.1d) and (3.1d′) for any fixed z, $z_j \in \Re_+$, $\sum_{j=1}^J z_j = 1$, (f) implies that

$$\vec{D}_i^0(y, x; g) = \min_{j=1, ..., J} \{\vec{D}_i^{zj}(y, x; g)\}.$$

Hence, for arbitrary z

$$\vec{D}_i^0(y, x; g) = \sup\left\{\min_{j=1, ..., J} \{\vec{D}_i^{zj}(y, x; g)\}\colon\right.$$
$$\left. z_j \in \Re_+, j=1, ..., J, \sum_{j=1}^J z_j = 1\right\}.$$

That (g′) follows from (g) can be shown by applying (c) and (c′) for a fixed element $\hat{z} \in L^*(y)$.

To prove (h) and (h′), first note that McFadden [15] shows that (h) implies that

$$D_i^0(y, x) = D_i^*(y, D_i^1(y, x), ..., D_i^J(y, x)).$$

Applying (1.7),

$$\begin{aligned}\vec{D}_i^0(y, x; g) &= \sup\{\beta \in \mathfrak{R}: D_i^*(y, D_i^1(y, x-\beta g), ..., D_i^J(y, x-\beta g)) \geqslant 1\}\\ &= \sup\{\beta \in \mathfrak{R}: D_i^*(y, z) \geqslant 1, D_i^k(y, x-\beta g) \geqslant z_k \geqslant 0, k=1, ..., J\}\end{aligned}$$

Because $D_i^k(y, x-\beta g) \geqslant z_k$ for all k we can now apply (3.1d) and (3.1d′) to obtain

$$\vec{D}_i^0(y, x; g) = \sup_{z \in L^*(y)} \{ \min_{j=1, ..., J} \{\vec{D}_i^{zj}(y, x; g)\}\}. \qquad \text{Q.E.D.}$$

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